ELSEVIER
Journal of Pure and Applied Algebra 161 (2001) 269–294
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Sums of powers in function fields

Ralph Berr[a], Franç oise Delon[b,*]

[a] *Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany*
[b] *CNRS-Université Paris 7, Case 7012, 2 Place Jussieu, 75251 Paris cedex 05, France*

Received 10 May 1999; received in revised form 28 February 2000
Communicated by M.-F. Roy

Abstract

When is a rational function on a smooth variety V whose values are sums of $2n$th powers, itself a sum of $2n$th powers in the field of rational functions? We investigate this property and some weaker form of it. The aim is to gain a better understanding of sums of powers in formally real function fields. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 12D15; 12J15; 14A10

0. Introduction

In this paper, we continue the investigations on Hilbert's 17th problem for sums of $2n$th powers started in [3]. We briefly recall the situation we are concerned with. Let K be a field of characteristic 0, V an affine irreducible variety over K with function field $K(V)$. Given $n \in \mathbb{N}$ we consider the problem whether the sums of $2n$th powers in $K(V)$ can be characterized by certain geometric conditions. To this end, we say that $V|K$ satisfies the property S_n (resp. T_n) if for all $f \in K[V]$

$$S_n : \ f \in \sum K(V)^{2n} \Leftrightarrow f(x) \in \sum K(x)^{2n} \quad \text{for every closed point } x \in V_{\mathrm{reg}},$$
$$T_n : \ f \in \sum K(V)^{2n} \Leftrightarrow f(x) \in \sum K^{2n} \quad \text{for every point } x \in V(K)_{\mathrm{reg}}.$$

Now let

$$S(V|K) := \{n \in \mathbb{N} \mid V|K \text{ satisfies } S_n\},$$

$$T(V|K) := \{n \in \mathbb{N} \mid V|K \text{ satisfies } T_n\}.$$

* Corresponding author.
E-mail address: delon@logique.jussieu.fr (F. Delon).

PII: S0022-4049(00)00113-4

In [3] basic properties of $S(V|K)$ and $T(V|K)$ were proven. For example, $S(V|K)$ is actually a multiplicative semigroup with 1 generated by a set of primes. If $T(V|K)$ is not empty the same holds for $T(V|K)$ [3, Theorem 1.8]. Moreover, these semigroups have been computed explicitly for generalized real closed fields and function fields over totally Archimedean fields.

It is one of the aims of this paper to develop new methods for the study of the semigroups $S(V|K)$ and $T(V|K)$. These methods apply to arbitrary ground fields K. Therefore, we start with general characterizations of the semigroup $S(V|K)$ both by model theoretical and valuation theoretical properties of the function field $K(V)$. In Section 2, the case is treated where the ground field K is a function field over some field k with tr.d.$(K|k) \geq 1$. We prove the crucial result that in this situation $S(V|K)=\mathbb{N}$ for all irreducible varieties V over K. Moreover, if tr.d.$(K|k) \geq 2$, then even $T(\mathbb{A}^d|K)=\mathbb{N}$ for all $d \in \mathbb{N}$. Section 3 is devoted to a glimpse on the independence problem. Let V, W be varieties over K. In all cases known so far, one has $S(V|K)=S(W|K)$ which led to the conjecture that $S(V|K)$ is independent from the variety V, i.e. $S(V|K)$ depends only on the ground field K. From the previous results, we infer some very first steps towards an affirmative answer to this independence-conjecture. For example, if K is any field and $d \in \mathbb{N}$ with $d \geq 2$ then

$$S(\mathbb{A}^d|K) = S(\mathbb{A}^2|K) \quad \text{and} \quad T(\mathbb{A}^d|K) = T(\mathbb{A}^2|K).$$

In the next step, we return to the "classical" situation, i.e. we consider the case of sums of squares. According to the general solution of Hilbert's 17th problem, we have $1 \in S(V|K)$ for all varieties V over K. Therefore, it remains to characterize the pairs $V|K$ which satisfy $1 \in T(V|K)$. So far such a characterization has only been known for $V = \mathbb{A}^d$. In Section 4, we solve the general case. For example, if V is a smooth variety over K then $1 \in T(V|K)$ if and only if $V(K)^{\geq}$ is dense in $\operatorname{Sper} K[V]$, where

$$V(K)^{\geq} = \{\alpha \in \operatorname{Sper} K[V] | K[V]/\operatorname{supp}(\alpha) = K\}$$

denotes the subspace of the "K-rational points" of $\operatorname{Sper} K[V]$. Finally, we treat the question which semigroups $M \subset \mathbb{N}$ are realized as some $T(V|K)$. In the case of $S(V|K)$ this problem was solved in [3]: Let $\mathbb{P}_0$ be any set of prime numbers and let $\langle \mathbb{P}_0 \rangle \subset \mathbb{N}$ be the semigroup with 1, generated by $\mathbb{P}_0$, then there exists a field K such that $S(V|K)= \langle \mathbb{P}_0 \rangle$ for all varieties V over K [3, Corollary 2.4]. But in the case of $T(V|K)$ so far only examples for $T(V|K)=\emptyset$, $T(V|K)=\{1\}$ and $T(V|K)=\mathbb{N}$ have been known. In the final section, we will show that for every set $\mathbb{P}_0$ of prime numbers with $2 \in \mathbb{P}_0$ there exists a field K such that $T(\mathbb{A}^d|K)=\langle \mathbb{P}_0 \rangle$ for all $d \in \mathbb{N}$.

0.1. Notations and conventions

All fields are assumed to be of characteristic 0. By a variety over K we mean an integral affine scheme $V = \operatorname{Spec} A$ of finite type over K. Given $x \in V$ and $f \in A$ we let $K(x)$ denote the residue field of x, $f(x)$ its residue class in $K(x)$ and V_{reg} the set of regular points of V.

Let V be a variety over K and let $L|K$ be any field extension. By definition, an L-rational point of V is a morphism $x:\operatorname{Spec} L \to \operatorname{Spec} A$ of schemes over K and we let $V(L)$ denote the set of L-rational points of V. Each $x \in V(L)$ is induced by a unique K-homomorphism $x^*: A \to L$. Given any $f \in A$ and $x \in V(L)$ we call $K(x):=\operatorname{quot}(x^*(A))$ the residue field of x and $f(x):=x^*(f)$ the value of f at x. Let $\bar{x} = \ker(x^*) \in \operatorname{Spec} A$. Then $K(x) \cong K(\bar{x})$ and we will identify $f(x)$ with $f(\bar{x})$ via this map. A point $x \in V(L)$ will be called regular if $x : \operatorname{Spec} L \to \operatorname{Spec} A$ is smooth. $V(L)_{\text{reg}}$ is the set of regular points of $V(L)$. Since $\operatorname{char}(K)=0$, $x \in V(L)$ is regular if and only if $\bar{x} \in V_{\text{reg}}$. In order to apply model theoretic techniques we will often fix a representation

$$A = K[X_1,\dots,X_n]/(f_1,\dots,f_k) = K[x_1,\dots,x_n],$$

where x_i denotes the residue class of X_i in A. Let $y \in V(L)$ and $y_i = y^*(x_i) \in L$ for $i = 1,\dots,n$. We will identify $V(L)$ via

$$V(L) \to \mathbb{A}^n(L): y \mapsto (y_1,\dots,y_n)$$

with the L-algebraic set

$$\{y \in \mathbb{A}^n(L) \mid f_1(y) = \cdots = f_k(y) = 0\}.$$

Let $d = \dim V$. Then $y \in V(L) \subset \mathbb{A}^n(L)$ is regular if and only if the Jacobian matrix $J(f_1,\dots,f_k)$ evaluated at y has rank $n-d$. Hence, there exists a first-order formula Φ^V_{reg} in the language of fields (with constants from K) such that for any extension field $L|K$ and any $y \in V(L)$ we have

$$y \in V(L)_{\text{reg}} \quad \Leftrightarrow \quad L \models \Phi^V_{\text{reg}}(y).$$

If the function field $K(V)$ of V is not real then $S(V|K) = \mathbb{N} = T(V|K)$ by [4, Corollary 1.2]. Therefore, we assume throughout this paper that all considered varieties V are real, i.e. the function field $K(V)$ is a real field.

Finally, let p be a prime number and $f \in K(V)$ a rational function on V with $f \notin \sum K(V)^{2p}$. By the expression "there is a closed point $x \in V_{\text{reg}}$ with $f(x) \notin \sum K(x)^{2p}$" we mean that there are $g \in K[V]$ and a closed point $x \in V_{\text{reg}}$ with $g^{2p}f \in K[V]$ and $(g^{2p}f)(x) \notin \sum K(x)^{2p}$. Hence, $p \in S(V|K)$ if and only if for every rational function $f \in K(V)$ with $f \notin \sum K(V)^{2p}$ there exists a closed point $x \in V_{\text{reg}}$ such that $f(x) \notin \sum K(x)^{2p}$.

1. Characterization of $S(V|K)$

In this section, we use both model theory and valuation theory to characterize $S(V|K)$. In the first step we will investigate $S(V|K)$ from the model theoretic point of view. By [4, Theorem 1.8], $S(V|K)$ is a semigroup with 1, generated by a set of primes. Thus, it is sufficient to find a necessary and sufficient condition for a prime p

to be in $S(V|K)$. Given any prime p and any field F, then

$$\sum F^{2p} = \bigcap P,$$

where P ranges over all orderings of F of level 1 or p [1, Satz 2.18]. We will show that $p \in S(V|K)$ if and only if

$$\sum K(V)^{2p} = \bigcap P,$$

where P ranges over a certain subclass of orderings of $K(V)$. For convenience, we recall the basic notions which will be used in the formulation of the precise statement. Details and further references can be found in [3,4,10].

Let L be any real field, p a prime and $\hat{\mathbb{Z}}_p$ the additive group of the p-adic integers. A homomorphism

$$\varphi : L^* \to \{1, -1\} \times \hat{\mathbb{Z}}_p$$

is called a *chain signature* of L if $\ker(\varphi)$ is a valuation fan. Equivalently, there are a total order $Q \subset L$, a valuation v of L compatible with Q and a homomorphism $\bar{\varphi} : v(L^*) \to \hat{\mathbb{Z}}_p$ such that

$$\varphi = \mathrm{sign}_Q \times (\bar{\varphi} \circ v),$$

see [10, Corollary 2]. Given a chain signature φ and $n \in \mathbb{N}$ we set

$$P_{p^n}(\varphi) := \varphi^{-1}(1 \times p^n \hat{\mathbb{Z}}_p) \cup \{0\}.$$

Note that $P_1(\varphi)$ is a total order of L. We call φ *p-primary* if there is some $n \in \mathbb{N}$ with $P_1(\varphi) \neq P_{p^n}(\varphi)$. Otherwise, we say that φ is a *trivial* chain signature.

Now let $F|K$ be a field extension and let φ be a chain signature of F. We let $(\hat{F}, \hat{\varphi})$ denote the real closure of (F, φ) and $\hat{K}$ the relative algebraic closure of K in $\hat{F}$. Recall that $(\hat{K}, \hat{\varphi}_{|\hat{K}})$ is the real closure of $(K, \varphi_{|K})$ [4, Proposition 1.20]. Moreover, if φ is a trivial chain signature, then $\hat{F}$ is just the usual real closure of F with respect to the total order $P_1(\varphi)$.

Finally, let $\mathscr{L}_p$ be the extension of the language of fields by a unary predicate P_{p^n} for each $n \in \mathbb{N}_0$. Thus, if φ is a chain signature of F, then $(F, (P_{p^n}(\varphi))_{n \in \mathbb{N}_0})$ is in a canonical way a $\mathscr{L}_p$-structure.

Definition 1.1. Let $F|K$ be a field extension. A chain signature φ of F is called K-admissible if $(\hat{K}, \hat{\varphi}_{|\hat{K}})$ is an elementary substructure of $(\hat{F}, \hat{\varphi})$ with respect to the language $\mathscr{L}_p$.

Note that every trivial chain signature of F is K-admissible since in this situation $\hat{F}$ and $\hat{K}$ are real closed fields in the sense of Artin–Schreier. Admissible p-primary chain signatures can be characterized as follows: Given any p-real closed field R we let $J(R)$ denote the largest valuation ring of R such that the corresponding residue field is real closed. Back to our situation above, we get from [4, Theorem 2.5]: If φ is a p-primary chain signature of K, then $\hat{K}$ is an elementary substructure of $\hat{F}$ if and only if

(1) $\hat{K}$ is p-real closed, and
(2) $J(\hat{F})\bigcap\hat{K}=J(\hat{K})$.

Let us return to the problem we are concerned with. Given any variety V over K it follows from [1, Satz 2.18] and [10, Proposition 6] that

$$\sum K(V)^{2p}=\bigcap P_p(\varphi),$$

where φ runs through the chain signatures of $K(V)$. If $p\in S(V|K)$ it is even sufficient to consider the K-admissible chain signatures, as we will show now.

Proposition 1.2. *Let V be a variety over K with function field F and let p be a prime. Then*

$$p\in S(V|K)\Leftrightarrow\sum F^{2p}=\bigcap P_p(\varphi),$$

where φ ranges over the K-admissible chain signatures of F.

Proof. "$\Rightarrow$": Let $f\in K[V]\setminus\sum F^{2p}$. Then, there exists a closed point $x\in V_{\mathrm{reg}}$ with $f(x)\notin\sum K(x)^{2p}$, as $p\in S(V|K)$. Since x is regular the evaluation map $e_x:K[V]\to K(x)$ extends to a place

$$\lambda:F\to K(x)\cup\infty.$$

As $f(x)\notin\sum K(x)^{2p}$ there is a chain signature ψ of $K(x)$ with $f(x)\notin P_p(\psi)$. By [10, Lemma 9], ψ can be lifted to a chain signature φ of F, i.e.

$$\varphi(y)=\psi(\lambda(y))$$

for all $y\in B_\lambda$, where B_λ denotes the valuation ring of λ. If ψ is trivial we let φ denote a lifting of ψ which is trivial as well. Since λ extends e_x, we have $K[V]\subset B_\lambda$. Hence, $\varphi(f)=\psi(\lambda(f))=\psi(f(x))$. Now $f(x)\notin P_p(\psi)$ implies $f\notin P_p(\varphi)$. Therefore, it remains to show that φ is K-admissible. If ψ is trivial then φ is trivial, hence K-admissible. So assume that ψ is p-primary. Let $(\hat{F},\hat{\varphi})$ be the real closure of (F,φ) and $\hat{K}$ the relative algebraic closure of K in $(\hat{F},\hat{\varphi})$. Since $\varphi_{|K}=\psi$, we see that $\hat{K}$ is a real closure of (K,ψ) [4, Proposition 1.20]). In particular, $\hat{K}$ is p-real closed. Now let λ' be a real extension of λ to $\hat{F}$ and let $B_{\lambda'}$ be the valuation ring of λ'. By [5, Proposition (1.1)], $B_{\lambda'}$ is Henselian. Hence, the residue field $\hat{F}_{\lambda'}$ of λ' is isomorphic to a relatively algebraically closed subfield of $\hat{F}$. Since $\hat{F}_{\lambda'}$ is algebraic over K, $\hat{F}_{\lambda'}$ is isomorphic to $\hat{K}$. Let $\lambda_0:\hat{K}\to R_0\cup\infty$ be the canonical surjective place associated with $J(\hat{K})$. Then R_0 is real closed. Now let

$$B:=\lambda'^{-1}(J(\hat{F}_{\lambda'}))$$

and let λ_0' be the place corresponding to B. Then λ_0' is an extension of λ_0 with real closed residue field R_0, i.e. $B\subset J(\hat{F})$. Thus, $J(\hat{K})\subset J(\hat{F})\cap\hat{K}$ as $B\cap\hat{K}=J(\hat{K})$. But from the definition of the rings $J(\hat{F})$, $J(\hat{K})$ we immediately get $J(\hat{F})\cap\hat{K}\subset J(\hat{K})$. Hence $J(\hat{K})=J(\hat{F})\cap\hat{K}$. Therefore, φ is a K-admissible chain signature of F.

"$\Leftarrow$": Let $f \in K[V] \setminus \sum F^{2p}$. We have to find a closed point $x \in V_{\mathrm{reg}}$ with $f(x) \notin \sum K(x)^{2p}$. We proceed as follows. By assumption there exists a K-admissible chain signature φ of F with $f \notin P_p(\varphi)$. We keep φ fixed. Next, assume

$$K[V] = K[X_1,\dots,X_n]/(f_1,\dots,f_k)$$

and consider the $\mathscr{L}_p$-formula

$$\Phi(Z) := \Phi^V_{\mathrm{reg}}(Z) \wedge \neg P_p(f(Z)).$$

Let $x = (x_1,\dots,x_n)$ be the image of the indeterminates $X_1,\dots,X_n$ in $K[V]$. By the choice of $\hat{\varphi}$ and Φ we have $(\hat{F},\hat{\varphi}) \models \Phi(x)$. Since $(\hat{K},\hat{\varphi}_{|\hat{K}})$ is an elementary substructure of $(\hat{F},\hat{\varphi})$ we get

$$(\hat{K},\hat{\varphi}_{|\hat{K}}) \models \exists y\, \Phi(y).$$

But this implies that there exists a closed point $y \in V_{\mathrm{reg}}$ with $f(y) \notin \sum K(y)^{2p}$. $\square$

In particular, this result provides a new proof of [3, Proposition 1.9]:

Corollary 1.3. *Let V, W be birationally equivalent varieties over K. Then*

$$S(V|K) = S(W|K).$$

So far, we have used the characterization of sums of $2n$th powers via orderings of higher level. There is another one in the framework of valuation theory. Let $x \in \sum K^2$. Then, we have [2, Theorem 1.9]:

$$x \in \sum K^{2n} \Leftrightarrow 2n | v(x) \quad \text{for all real valuations } v \text{ of } K.$$

Now let $\mathscr{V}$ be a set of real valuations of K. We call $\mathscr{V}$ a *p-deciding family* for K if for all $x \in \sum K^2$:

$$x \in \sum K^{2p} \Leftrightarrow 2p \mid v(x) \quad \text{for all } v \in \mathscr{V}.$$

In the next step, we will show that the semigroup $S(V|K)$ can be described by the existence of certain deciding families for the function field $K(V)$.

Let $F|K$ be a function field. Given a prime p we let $\mathscr{V}_p(F)$ denote the set of real valuations v of F which satisfy the following conditions:

(V1) The residue field F_v admits an Archimedean ordering.
(V2) $v(K) \not\subset pv(F)$.
(V3) The relative divisible hull $v(K)^{\mathrm{div}}$ of $v(K)$ in $v(F)$ is a convex subgroup.

Theorem 1.4. *Let V be a variety over K with function field F and let p be a prime. Then*

$$p \in S(V|K) \Leftrightarrow \mathscr{V}_p(F) \text{ is a } p\text{-deciding family for } F.$$

Proof. Let $f \in F$ with $f \in \sum F^2 \setminus \sum F^{2p}$. We first assume $p \in S(V|K)$. Then, there exists a closed point $x \in V_{\text{reg}}$ with $f(x) \notin \sum K(x)^{2p}$. Let $e_x : K[V] \to K(x)$ be the corresponding evaluation map. Since x is regular, e_x extends to a place

$$\lambda : F \to K(x) \cup \infty.$$

Moreover, since $\lambda(f) = f(x) \notin \sum K(x)^{2p}$ there exists a real valuation w of $K(x)$ such that $w(f(x)) \notin 2pw(K(x))$. We may assume w.l.o.g. that the residue field of w admits an Archimedean ordering. Let A_w be the valuation ring of w and let v be the valuation corresponding to $\lambda^{-1}(A_w)$. Then, $v \in \mathscr{V}_p(F)$ and $w(f(x)) \notin 2pw(K(x))$ implies $v(f) \notin 2pv(F)$. Next, assume that $\mathscr{V}_p(F)$ is a p-deciding family for F and let $f \in F$ be as before. By Proposition 1.2 it is sufficient to show that there is a K-admissible chain signature φ of F with $f \notin P_p(\varphi)$. In the first step, we prove that there are $v \in \mathscr{V}_p(F)$ and a homomorphism $\bar{\varphi} : v(K) \to \hat{\mathbb{Z}}_p$ such that
(1) $\bar{\varphi}(v(f)) \notin p\hat{\mathbb{Z}}_p$,
(2) $\bar{\varphi}(v(K)) \not\subset p\hat{\mathbb{Z}}_p$.

By assumption there is $v \in \mathscr{V}_p(F)$ with $v(f) \notin 2pv(F)$. Let $\bar{\varphi}_1 : \mathbb{Z}v(f) \hookrightarrow \hat{\mathbb{Z}}_p$ be the canonical embedding. Pick any $\gamma \in v(K) \setminus pv(F)$. First assume $v(f) - \gamma \in pv(F)$. Since $\mathbb{Z}v(f)$ is a p-pure subgroup of $v(F)$ and since $\hat{\mathbb{Z}}_p$ is p-pure injective, $\bar{\varphi}_1$ extends to a homomorphism $\bar{\varphi} : v(F) \to \hat{\mathbb{Z}}_p$. Then $\bar{\varphi}(v(f)), \bar{\varphi}(\gamma) \notin p\hat{\mathbb{Z}}_p$. So assume $v(f) - \gamma \notin pv(F)$. Then $\mathbb{Z}v(f)$ is a p-pure subgroup of

$$G = v(F)\left[\left.\frac{v(f)-\gamma}{p^n}\right| n \in \mathbb{N}\right].$$

Thus, $\bar{\varphi}_1$ extends to a homomorphism $\bar{\varphi}_G : G \to \hat{\mathbb{Z}}_p$ and the construction of G shows $\bar{\varphi}(v(f)), \bar{\varphi}(\gamma) \notin p\hat{\mathbb{Z}}_p$. Hence the restriction $\bar{\varphi}$ of $\bar{\varphi}_G$ to $v(F)$ is a homomorphism of the desired kind. Now let $Q \subset F$ be a total order compatible with v and set

$$\varphi := \operatorname{sign}_Q \times (\bar{\varphi} \circ v).$$

Note that $f \notin P_p(\varphi)$. Thus, it remains to show that φ is a K-admissible chain signature of F. Let $\hat{F}$ be the real closure of (F, φ) and let $\hat{K}$ be the relative algebraic closure of K in $\hat{F}$. Then, $\hat{K}$ is p-real closed by (2). Let $\hat{v}$ be a real extension of v to $\hat{F}$. Since $\hat{v}(\hat{K})$ is a convex subgroup of $\hat{v}(\hat{F})$ we get $J(\hat{F}) \cap \hat{K} = J(\hat{K})$, hence $\hat{K} \prec \hat{F}$. This shows that φ is a K-admissible chain signature of F. □

2. Varieties over function fields

Let K be a function field over k with tr.d.$(K|k) \geq 1$. In this section, we will prove that $S(V|K) = \mathbb{N}$ for every variety over K. This result is essential for the study of the semigroup $S(V|K)$ in the general situation as will become apparent later on.

Let R be a field which is either real closed or p-real closed. Then R admits a Henselian valuation w with Archimedean real closed residue field (cf. [4, Theorem 1.4]) and we let $A(R) \subset R$ denote the valuation ring of w. Let $Q \subset R$ be a total order. Then w is compatible with Q and R is real closed with respect to a chain signature

$\varphi = \mathrm{sign}_Q \times (\bar{\varphi} \circ w)$ (R is real closed iff $\bar{\varphi}$ is the trivial map). Now let X be transcendental over R. Extend Q to a total order $P \subset R(X)$ such that the value group of the natural valuation w_P of P has the form

$$\mathbb{Z} \cdot w_P(X) \oplus w_P(R)$$

with $\mathbb{Z} \cdot w_P(X)$ a convex subgroup of $w_P(R)$. Let $i : \mathbb{Z} \to \hat{\mathbb{Z}}_p$ be the canonical embedding. Then,

$$\psi := \mathrm{sign}_P \times ((i \oplus \bar{\varphi}) \circ w_P)$$

is a p-primary chain signature of $R(X)$ extending φ. We let R_p denote the real closure of $(R(X), \psi)$. Note that R_p does not depend on the choice of Q and the construction of R_p implies $A(R_p) = J(R_p)$. Moreover, if $g : R \to L$ is a homomorphism and L is real closed or p-real closed, then g extends in a canonical way to a homomorphism $g_p : R_p \to L_p$. Hence, we may regard R_p as a subfield of L_p. Then $A(L_p) = J(L_p)$ implies $J(L_p) \cap R_p = A(R_p) = J(R_p)$. Hence $R_p \prec L_p$.

The next result generalizes [3, Theorem 3.1] where the corresponding statement was proven for function fields over totally Archimedean fields.

Theorem 2.1. *Let K be a function field over k with* tr.d.$(K|k) \geq 1$. *Then, for every irreducible variety over K:*

$$S(V|K) = \mathbb{N}.$$

Proof. Let F be the function field of V. Since $K|k$ is a function field, there exists $k' \subset K$ such that tr.d.$(K|k') = 1$. Therefore, we may assume w.l.o.g. that tr.d.$(K|k) = 1$. Thus,

$$K = k(X)[Y]/(g) = k(X, y)$$

for some irreducible polynomial g which is nonconstant with respect to Y. Next fix a representation

$$F = K(T)[U]/(h) = K(T, u) = k(X, T, y, u),$$

where $T = (T_1, \ldots, T_n)$ is a tuple of indeterminates and $h \in k[X, Y, T, U]$ is an irreducible polynomial which is nonconstant with respect to U. By Corollary 1.3 we may assume w.l.o.g.

$$V = \mathrm{Spec}\, K[T, U]/(h).$$

Let $f \in \sum F^2 \setminus \sum F^{2p}$. Then there exists a p-real closure $R \supset F$ with $f \notin R^{2p}$. Let $\tilde{k}$ be the relative algebraic closure of k in R. Then, the following diagram commutes:

$$\begin{array}{ccccc} F & - & R & - & R_p \\ | & & | & & | \\ k & - & \tilde{k} & - & \tilde{k}_p \end{array}.$$

By definition of the fields $R_p, \tilde{k}_p$ we have $J(R_p) = A(R_p)$ and $J(\tilde{k}_p) = A(\tilde{k}_p)$. Thus $\tilde{k}_p \prec R_p$ by [4, Theorem 2.3]. Now let $\Phi(x, y, t, u)$ be the formula

$$g(x, y) = 0 \ \wedge \ h(x, y, t, u) = 0 \wedge$$
$$\operatorname{rank}\left(\begin{pmatrix} g_y & 0 \\ h_y & h_u \end{pmatrix}(x, y, t, u)\right) = 2 \wedge \neg P_p(f(x, y, t, u)).$$

Then $R_p \models \Phi(X, y, T, u)$. Hence,

$$\tilde{k}_p \models \exists x_0, y_0, t_0, u_0\, \Phi(x_0, y_0, t_0, u_0).$$

Let $Q \subset \tilde{k}_p$ be a total order. By the implicit function theorem for p-real closed fields there exists $\varepsilon \in Q \setminus \{0\}$ such that for all $z \in\,]x_0 - \varepsilon, x_0 + \varepsilon[=: I$ there are $z_1, z_2^1, \ldots, z_2^n, z_3 \in \tilde{k}_p$ with $\tilde{k}_p \models \Phi(z, z_1, z_2, z_3)$. Since I contains an element which is transcendental over k, we may assume w.l.o.g. that x_0 is transcendental over k. Hence,

$$K = k(X, y) \simeq k(x_0, y_0) \subset \tilde{k}_p.$$

Therefore, we consider K as a subfield of $\tilde{k}_p$. Then $\tilde{k}_p$ is algebraic over K and $(t_0, u_0) \in V(\tilde{k}_p)$. From the definition of Φ we see

$$(t_0, u_0) \in V(\tilde{k}_p)_{\text{reg}} \quad \text{and} \quad f(t_0, u_0) = f(x_0, y_0, t_0, u_0) \notin \tilde{k}_p^{2p}.$$

Hence, $f(x_0, y_0, t_0, u_0) \notin \sum K(t_0, u_0)^{2p}$. $\square$

In view of Theorem 2.1 the next result can be proved in the same manner as [3, Theorem 3.4], where varieties over totally Archimedean fields were considered. For convenience we restate the proof.

Proposition 2.2. *Let K be any field and let V be a smooth variety over K with* $\dim V \geq 2$. *Then for $f \in K[V]$ and $n \in \mathbb{N}$ the following statement holds*:

$$f \in \sum K(V)^{2n} \Leftrightarrow \textit{for every irreducible curve } C \subset V : f_{|C} \in \sum K(C)^{2n}.$$

Proof. Let $C \subset V$ be an irreducible curve. Then $K[C] \cong K[V]/\wp$ for some $\wp \in \operatorname{Spec} K[V]$. Hence we have a canonical homomorphism $\pi : K[V] \to K(C)$. Since $K[V]$ is regular, π extends to a real place

$$\lambda : K(V) \to K(C) \cup \infty.$$

Let $f \in \sum K(V)^{2n}$. Then $f_{|C} = \pi(f) = \lambda(f) \in \sum K(C)^{2n}$, as λ is real. For the converse direction fix a representation

$$K[V] = K[x_1, \ldots, x_m] = K[X_1, \ldots, X_m]/\wp.$$

We may assume w.l.o.g. that x_1 is not algebraic over K. Let $S = K[X_1] \setminus \{0\}$. Then

$$K(X_1)[X_2, \ldots, X_m]/\wp_S \cong K[V]_S.$$

Let $W = \operatorname{Spec} K[V]_S$. Then W is a variety over $K(X_1)$. Let $f \in K[V]$. We consider f as a regular function on W via the canonical embedding

$$\varphi : K[V] \to K[V]_S = K(X_1)[W].$$

By Theorem 2.1 we have

$$(*)\quad f \in \sum K(V)^{2n} \Leftrightarrow f(z) \in \sum K(X_1)(z)^{2n} \quad \text{for all closed points } z \in W_{\mathrm{reg}}.$$

The embedding φ induces a morphism

$$\varphi^* : W = \operatorname{Spec} K[V]_S \to \operatorname{Spec} K[V] = V.$$

Given a closed point $z \in W$, then $\varphi^*(z)$ is the kernel of the homomorphism

$$\varphi_z : K[V] \to K[V]_S \to K(X_1)(z).$$

Hence the point z corresponds to some $\wp \in \operatorname{Spec} K[V]$ with $\dim \wp = 1$. Now the claim follows from $(*)$. □

Before stating further consequences of Theorem 2.1, we mention the following simple fact:

Lemma 2.3. *Let K be any field and let V be a variety over K. Given $d \in \mathbb{N}$ then*

$$S(V \times \mathbb{A}^d|K) \subset S(V|K) \quad \text{and} \quad T(V \times \mathbb{A}^d|K) \subset T(V|K).$$

Proof. Let $n \in \mathbb{N}$ and $f \in K(V) \setminus \sum K(V)^{2n}$. Then one readily verifies $f \notin \sum K(V)(X_1,\dots,X_d)^{2n} = \sum K(V \times \mathbb{A}^d)^{2n}$ which implies $S(V \times \mathbb{A}^d|K) \subset S(V|K)$ and $T(V \times \mathbb{A}^d|K) \subset T(V|K)$. □

In particular, we have the sequences

$$S(\mathbb{A}^1|K) \supset S(\mathbb{A}^2|K) \supset S(\mathbb{A}^3|K) \supset \dots,$$

$$T(\mathbb{A}^1|K) \supset T(\mathbb{A}^2|K) \supset T(\mathbb{A}^3|K) \supset \dots$$

In view of Theorem 2.1 we get the stronger result:

Corollary 2.4. *Let K be any field and let $d \in \mathbb{N}$ with $2 \le d$. Then*

$$S(\mathbb{A}^d\,|\,K) = S(\mathbb{A}^2\,|\,K) \subset S(\mathbb{A}^1|K).$$

Proof. It remains to show $S(\mathbb{A}^2|K) \subset S(\mathbb{A}^d|K)$ for $d \ge 3$. By Theorem 2.1 we know $S(\mathbb{A}^{d-1}\,|\,K(X_1)) = \mathbb{N}$. Hence,

$$(*)\quad T(\mathbb{A}^{d-2}\,|\,K(X_1,X_2)) = \mathbb{N}$$

by [3, Proposition 1.11]. Now let $p \in S(\mathbb{A}^2\,|\,K)$ and

$$f \in K(X_1,\dots,X_n) \setminus \sum K(X_1,\dots,X_n)^{2p}.$$

According to $(*)$ we find $g_3,\dots,g_d \in K(X_1,X_2)$ such that

$$f(X_1,X_2,g_3(X_1,X_2),\dots,g_d(X_1,X_2)) \notin \sum K(X_1,X_2)^{2p}.$$

By assumption $p \in S(\mathbb{A}^2 \mid K)$. Hence there exist z_1, z_2 in the algebraic closure C of K with

$$f(z_1, z_2, g_3(z_1, z_2), \ldots, g_d(z_1, z_2)) \notin \sum K(z_1, z_2)^{2p}.$$

Therefore, $S(\mathbb{A}^2 \mid K) \subset S(\mathbb{A}^d \mid K)$. $\square$

In Section 3, we will see that the same statement holds for $T(\mathbb{A}^d|K)$. In the proof of Corollary 2.4 we have used beside Theorem 2.1 the following result [3, Proposition 1.11]: if X is transcendental over K, then

$$S(\mathbb{A}^{d+1} \mid K) \subset T(\mathbb{A}^d \mid K(X))$$

for every $d \in \mathbb{N}$. In order to obtain stronger versions of Corollary 2.4 we need a generalization of this result. To this end we consider the following set: given any field K let

$$S(K) := \{n \in \mathbb{N} \mid n \in S(V|K) \text{ for all varieties } V \text{ over } K\}.$$

By [3, Theorem 1.8], $S(V|K)$ is a multiplicative semigroup with 1, generated by a set of primes. Hence

Corollary 2.5. *$S(K)$ is a multiplicative semigroup with* 1, *generated by a set of primes.*

The next result generalizes [3, Theorem 1.12].

Theorem 2.6. *Let K be a function field over k with* $\text{tr.d.}(K|k) \geq 1$. *Then, for all $d \in \mathbb{N}$*

$$S(k) \subset T(\mathbb{A}^d|K).$$

Proof. By [3, Theorem 1.12] we know $1 \in T(\mathbb{A}^d \mid K)$. So let $p \in S(k)$ be a prime number and let

$$f \in \sum K(Z_1, \ldots, Z_d)^2 \setminus \sum K(Z_1, \ldots, Z_d)^{2p}.$$

We have to show that there is $z \in \mathbb{A}^d(K)$ with $f(z) \notin \sum K^{2p}$. As in the proof of Theorem 2.1 we may assume $\text{tr.d.}(K|k) = 1$, i.e.

$$K = k(X)[Y]/(g)$$

for some irreducible polynomial $g \in k[X, Y]$ which is nonconstant relative to Y. Let

$$V = \operatorname{Spec} k[X, Y]/(g).$$

We may assume w.l.o.g.

$$f \in k[X, Y, Z_1, \ldots, Z_d]/(g) = k[X, y, Z_1, \ldots, Z_d] = k[V \times \mathbb{A}^d].$$

Then, we can regard f as a regular function on $V \times \mathbb{A}^d$. Let C be the algebraic closure of K. By Theorem 2.1 we know $p \in S(\mathbb{A}^d \mid K)$. Hence, there is $z \in \mathbb{A}^d(C)$ with

$$f(X, y, z) \notin \sum K(z)^{2p}.$$

Consequently, there is a p-real closed algebraic extension $R \supset K(z)$ with $f(X,y,z) \notin R^{2p}$. Note that

$$(X,y) \in V(R)_{\mathrm{reg}} \quad \text{and} \quad \frac{\partial g}{\partial Y}(X,y) \neq 0.$$

Let $Q \subset R$ be any total order. By the implicit function theorem for Henselian-valued fields there exist some open interval $I \subset R$ containing X and a function $\varphi : I \to R$ such that for all $x' \in I$:

$$(x', \varphi(x')) \in V(R)_{\mathrm{reg}} \quad \text{and} \quad f(x', \varphi(x'), z) \notin R^{2p}.$$

Let $Q' = K \cap Q$. Then we get an extension $(K, Q') \subset (R, Q)$ of ordered fields. By [3, Lemma 1.10] there are $x_0 \in I$ and polynomials $g_1 \ldots, g_d \in K[T]$ such that $z_i = g_i(x_0)$. Hence,

$$(*) \quad f(x_0, \varphi(x_0), g_1(x_0), \ldots, g_d(x_0)) \notin \sum R^{2p}.$$

Moreover, it follows easily from the proof of [3, Lemma 1.10] that we can assume in addition that x_0 is transcendental over k. But then there is a canonical k-isomorphism

$$\psi : k(x_0, \varphi(x_0)) \to K = k(X, y).$$

Hence, it follows from $(*)$ that there are $z_1', \ldots, z_d' \in K$ with $f(X, y, z_1', \ldots, z_d') \notin \sum K^{2p}$. □

This result has the following remarkable consequence.

Corollary 2.7. *Let K be a function field over k with* tr.d.$(K|k) \geq 2$. *Then, for any $d \in \mathbb{N}$*

$$T(\mathbb{A}^d \,|\, K) = \mathbb{N}.$$

Proof. Pick an intermediate field $k \subset F \subset K$ with tr.d.$(F|k) = 1$. Then Theorem 2.1 shows $S(F) = \mathbb{N}$. Hence $T(\mathbb{A}^d|K) = \mathbb{N}$ by Theorem 2.6. □

In Proposition 5.4 we will see that in general, the assumption tr.d.$(K|k) \geq 2$ is indeed necessary: there are function fields $K|k$ of transcendence degree 1 with $T(\mathbb{A}^d|K) = \{1\}$ for all $d \in \mathbb{N}$.

3. Some independence results

In this section, we will draw several consequences from the previous results. They can be regarded as the very first steps towards an affirmative answer to the following independence-conjecture.

Conjecture. *Given any field K and varieties V, W over K, then*

$$S(V|K) = S(W|K).$$

In other words, $S(V|K) = S(K)$ for all varieties over K.

Proposition 3.1. *Let K be an arbitrary field and let V be a variety over K with $dim(V) \geq 2$. Then for all $d \in \mathbb{N}$*

$$S(V|K) = S(V \times \mathbb{A}^d|K) \quad \textit{and} \quad T(V|K) = T(V \times \mathbb{A}^d|K).$$

Proof. We fix a representation

$$K[V] = K[x_1, \ldots, x_m] = K[X_1, \ldots, X_m]/(g_1, \ldots, g_l).$$

By Lemma 2.3 we know $S(V \times \mathbb{A}^d|K) \subset S(V|K)$. So let $p \in S(V|K)$ and

$$f \in K[x_1, \ldots, x_m, Y_1, \ldots, Y_d] \Big\backslash \sum K(x_1, \ldots, x_m, Y_1, \ldots, Y_d)^{2p} .$$

By Corollary 2.7 we find $h_1, \ldots, h_d \in K(V)$ such that

$$f(x_1, \ldots, x_m, h_1, \ldots, h_d) \notin \sum K(V)^{2p}.$$

By assumption $p \in S(V|K)$. Hence, there exists $z \in V_{\mathrm{reg}}$ with

$$f(z, h_1(z), \ldots, h_d(z)) \notin \sum K(z)^{2p}.$$

Thus, $S(V|K) \subset S(V \times \mathbb{A}^d|K)$. The second statement can be proved in the same way. □

As an immediate consequence we get the following analog to Corollary 2.4.

Corollary 3.2. *Let K be any field and let $d \in \mathbb{N}$ with $2 \leq d$. Then*

$$T(\mathbb{A}^d|K) = T(\mathbb{A}^2|K) \subset T(\mathbb{A}^1|K).$$

In order to obtain a similar result for arbitrary varieties over K we first mention the following fact.

Lemma 3.3. *Let V be a variety over K and assume $T(V|K) \neq \emptyset$. Then, $V(K)_{\mathrm{reg}}$ is Zariski-dense in V.*

Proof. Assume that $V(K)_{\mathrm{reg}}$ is not Zariski-dense in V. Then there is a regular function $0 \neq f \in K[V]$ which vanishes on $V(K)_{\mathrm{reg}}$. Pick any $g \in K[V]$ with $g \notin \sum K(V)^2$. Then $gf^2 \notin \sum K(V)^2$ as well but gf^2 vanishes on $V(K)_{\mathrm{reg}}$. Hence, $1 \notin T(V|K)$. Now [3, Proposition 1.6] shows $T(V|K) = \emptyset$. □

Corollary 3.4. *Let V be a variety over K with $\dim(V) \geq 2$. Then*

$$T(V|K) \subset T(\mathbb{A}^2|K).$$

Proof. The claim is trivial if $T(V|K) = \emptyset$. So assume that $T(V|K)$ is not empty. Let $p \in T(V|K)$ and

$$f \in K(T_1, T_2) \Big\backslash \sum K(T_1, T_2)^{2p} .$$

By Lemma 3.3 there exists $x \in V(K)_{\mathrm{reg}}$. The evaluation map $e_x : K[V] \to K$ extends to a place

$$\lambda_x : K(V) \to K \cup \infty.$$

Moreover, λ_x extends canonically to a place

$$\lambda : K(V)(T_1,T_2) \to K(T_1,T_2) \cup \infty$$

with $\lambda(T_i)=T_i$. Then $\lambda(f)=f$. Hence, $f \notin \sum K(V)(T_1,T_1)^{2p}$ as λ is real. By Corollary 2.7 we find $h_1,h_2 \in K(V)$ with

$$f(h_1,h_2) \notin \sum K(V)^{2p}.$$

Now $p \in T(V|K)$ implies $p \in T(\mathbb{A}^2|K)$. $\square$

Exactly, the same argument proves the following statement.

Corollary 3.5. *Let V be a variety over K with $\dim(V) \geq 2$ and assume that $V(K)_{\mathrm{reg}}$ is not empty. Then $S(V|K) \subset S(\mathbb{A}^2|K)$.*

Finally, we use Theorems 2.1 and 1.4 to give a new characterization of the semigroup $S(K)$.

Proposition 3.6. *Let K be any field. Then*

$$S(K) = \{n \in \mathbb{N} \mid n \in S(C|K) \text{ for all curves } C \text{ over } K\}.$$

Proof. By definition, $S(K)$ is contained in the set on the right-hand side. So let p be a prime number and assume $p \in S(C|K)$ for all curves C over K. Given any variety V over K with $\dim V \geq 2$ we have to show $p \in S(V|K)$. Let $F = K(V)$ and

$$K[V] = K[x_1,\dots,x_m] = K[X_1,\dots,X_m]/(g_1,\dots,g_l).$$

We may assume w.l.o.g. that the residue class x_1 of X_1 is transcendental over K. Let

$$W = \operatorname{Spec} K(X_1)[X_2,\dots,X_m]/(g_1,\dots,g_l).$$

Now pick any $f \in F \setminus \sum F^{2p}$. Since by Theorem 2.1 $S(W \mid K(x_1)) = \mathbb{N}$, there exists a closed point $z \in W_{\mathrm{reg}}$ with

$$f(x_1,z) \notin \sum K(x_1)(z)^{2p}.$$

Let $\lambda : F \to K(x_1)(z) \cup \infty$ be the extension of the evaluation homomorphism $e_z : K(x_1)[W] \to K(x_1)(z)$. Next, let C be any model of $K(x_1)(z) \mid K$. Then C is a curve and by assumption we have $p \in S(C \mid K)$. Hence, there exists by Theorem 1.4 a real valuation w of $K(C) = K(x_1)(z)$ such that $w \in \mathscr{V}_p(K(C))$ and $w(f(x_1,z)) \notin 2pw(K(C)^*)$. Let $A \subset K(C)$ be the valuation ring of w and let v be the canonical valuation of F corresponding to $\lambda^{-1}(A)$. Then, $v(f) \notin 2pv(F^*)$. Since, $w(K(C)^*)$ corresponds to a convex subgroup of $v(F^*)$ and since the residue fields of v and w coincide, we get $v \in \mathscr{V}_p(F)$. Hence, Theorem 1.4 implies $p \in S(V|K)$. $\square$

Corollary 3.7. *Let K be any field and assume $S(K) = S(C|K)$ for all curves over K. Then $S(K) = S(V|K)$ for all varieties V over K with $V(K)_{\text{reg}} \neq \emptyset$.*

Proof. Let V be a variety over K with $\dim V \geq 2$ and $V(K)_{\text{reg}} \neq \emptyset$. By definition $S(K) \subset S(V|K)$ and Corollary 3.5, Corollary 2.4 show $S(V|K) \subset S(\mathbb{A}^2|K) \subset S(\mathbb{A}^1|K)$. By assumption $S(K) = S(\mathbb{A}^1|K)$. Hence $S(V|K) = S(K)$. □

4. The classical case – sums of squares

Let V be any irreducible variety over K. Then $1 \in S(V|K)$ according to the general solution of Hilbert's 17th problem. In contrast to this, we may have $T(V|K) = \emptyset$.

Example 4.1. Let $K = \mathbb{R}((X))$. Then $T(V|K) = \emptyset$ for all varieties over K (see [4, Theorem 2.6]).

It is the goal of this section to characterize the pairs $V|K$ which satisfy $T(V|K) \neq \emptyset$. By [3, Proposition 1.6] we know that $T(V|K)$ is not empty if and only if $1 \in T(V|K)$. Therefore, it is sufficient to determine the pairs $V|K$ with $1 \in T(V|K)$.

Let us recall some known results on this problem. If K is a real closed field then $1 \in T(V|K)$ for all varieties over K by the solution of Hilbert's 17th problem. Next, assume that $\sum K^2$ is already a total order and let R be the real closure of $(K, \sum K^2)$. Then it follows from McKenna's result [8] that $1 \in T(\mathbb{A}^d|K)$ if and only if K is dense in R. In 1979, Prestel generalized this result: if K admits only finitely many total orders, then $1 \in T(\mathbb{A}^d|K)$ if and only if K is dense in every real closure of K [9]. Note, if K is an arbitrary field then in general this equivalence does not hold.

Example 4.2. Let K be any function field over $\mathbb{R}$ with $\text{tr.d.}(K|\mathbb{R}) \geq 1$. Then, $1 \in T(\mathbb{A}^d|K)$ by Theorem 2.6. But K is not dense in any of its real closures.

In 1988, Zeng Guangxin solved the problem for affine spaces over arbitrary fields. He calls K locally dense, if for every finite real extension $L \supset K$ and all $a, b \in L$ there is a total order $Q \subset L$ such that the open interval $(a,b)_Q \subset L$ with respect to Q contains a point $x \in K$. Now, given any $d \in \mathbb{N}$, then $1 \in T(\mathbb{A}^d|K)$ if and only if K is locally dense [11, Theorem 3]. In this section we will prove in a similar manner a corresponding result for arbitrary varieties. Using the real spectrum of a ring we will eventually obtain a purely geometric description of the pairs $V|K$ which satisfy $1 \in T(V|K)$. In particular, we will get a new characterization of locally dense fields which is more convenient from the geometric point of view.

Let A be any commutative ring and let $\operatorname{Sper} A$ be the real spectrum of A (see [6]). Given $\alpha \in \operatorname{Sper} A$ we set $\operatorname{supp}(\alpha) = \alpha \cap -\alpha$ and let $k(\alpha)$ denote the real closure of the quotient field of $A/\operatorname{supp}(\alpha)$ with respect to the total order induced by α. In particular, if A is a field, then $k(\alpha)$ is just the real closure of (A, α). Finally we fix an algebraic closure $\hat{K}$ of our ground field K.

Lemma 4.3. *Let $f \in K[X_1,\dots,X_d]$ and $a \in \mathbb{A}^d(\hat{K})$ such that $L := K(a)$ is real. Given $\varepsilon \in K^*$ there is $\delta \in K^*$ such that for all $\alpha \in \operatorname{Sper} L$:*

$$k(\alpha) \models \forall x \in \mathbb{A}^d(k(\alpha)) \colon ||a-x|| < \delta^2 \Rightarrow |f(a) - f(x)| < \varepsilon^2.$$

Proof. Given $\delta \in K^*$ we set

$$C(\delta) := \{\alpha \in \operatorname{Sper} L \mid k(\alpha) \models \forall x \in \mathbb{A}^d(k(\alpha)) \colon ||a-x|| < \delta^2 \Rightarrow |f(a) - f(x)| < \varepsilon^2\}.$$

Then, $C(\delta)$ is a constructible subset of $\operatorname{Sper} L$. Let $\alpha \in \operatorname{Sper} L$. Since K is coinitial in $k(\alpha)$ we find $\delta_\alpha \in K^*$ such that $\alpha \in C(\delta_\alpha)$. By the compactness of $\operatorname{Sper} L$ there are finitely many $\delta_1,\dots,\delta_k \in K^*$ such that

$$\operatorname{Sper} L = \bigcup_{i=1}^{k} C(\delta_i).$$

Now let

$$\delta := \prod_{i=1}^{k} \frac{\delta_i^2}{1+\delta_i^2} \in K^*.$$

Then $\operatorname{Sper} L = C(\delta)$. $\square$

As an immediate consequence we get the following fact which can also be found in [11].

Corollary 4.4. *Let $f \in K[X_1,\dots,X_d]$ and $a \in \mathbb{A}^d(\hat{K})$ such that $L := K(a)$ is real and $f(a) \neq 0$. Then, there is $\delta \in K^*$ such that for all $\alpha \in \operatorname{Sper} L$:*

$$k(\alpha) \models \forall x \in \mathbb{A}^d(k(\alpha)) \colon ||a-x|| < \delta^2 \Rightarrow f(a) \cdot f(x) > 0.$$

We are now prepared to prove the characterization mentioned above.

Proposition 4.5. *Let V be a variety over K. Then the following statements are equivalent*:

(1) $1 \in T(V|K)$.
(2) $1 \in T(V \times \mathbb{A}^1|K)$.
(3) *Let $f \in K[V \times \mathbb{A}^1]$. If there are a real closure R of K and $z \in V(R)_{\mathrm{reg}} \times \mathbb{A}^1(R)$ with $f(z) = 0$, then for every $\varepsilon \in K^*$ there are $y \in V(K)_{\mathrm{reg}} \times \mathbb{A}^1(K)$ and a total order $Q \subset K$ such that $f(y)^2 <_Q \varepsilon^2$.*
(4) *Given a real closure R of K, $z \in V(R)_{\mathrm{reg}} \times \mathbb{A}^1(R)$ and $\varepsilon \in K^*$, there are $y \in V(K)_{\mathrm{reg}} \times \mathbb{A}^1(K)$ and a total order $Q \subset K(z)$ such that $||z-y|| <_Q \varepsilon^2$.*

Proof. Let us fix representations

$$K[V] = K[x_1,\dots,x_m] = K[X_1,\dots,X_m]/(g_1,\dots,g_l)$$

and

$$K[V \times \mathbb{A}^1] = K[x_1,\dots,x_m,T].$$

We will prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). So assume $1 \in T(V|K)$ and let

$$f \in K[V \times \mathbb{A}^1] \Big\backslash \sum K(V \times \mathbb{A}^1)^2 .$$

By Theorem 2.6 we have $1 \in S(K) \subset T(\mathbb{A}^1 \,|\, K(V))$. Hence, there is some $h \in K(V)$ such that

$$f(x, h(x)) \notin \sum K(V)^2,$$

where $x = (x_1, \ldots, x_m)$. Since $1 \in T(V \,|\, K)$ we find $z \in V(K)_{\text{reg}}$ with

$$f(z, h(z)) \notin \sum K^2.$$

Thus, $1 \in T(V \times \mathbb{A}^1 | K)$.

(2) $\Rightarrow$ (3): Let f, R and $z \in V(R)_{\text{reg}} \times \mathbb{A}^1(R)$ be as in (3). Let $\varepsilon \in K^*$. From $f(z) = 0$ we get $f^2 - \varepsilon^2 \notin \sum K(V \times \mathbb{A}^1)^2$. By (2) there is some $y \in V(K)_{\text{reg}} \times \mathbb{A}^1(K)$ with $f^2(y) - \varepsilon^2 \notin \sum K^2$ which just means that there is a total order $Q \subset K$ with $f^2(y) <_Q \varepsilon^2$.

(3) $\Rightarrow$ (4): Given any $h \in K[V \times \mathbb{A}^1]$ and $\alpha \in \operatorname{Sper} K[V \times \mathbb{A}^1]$ we let $h(\alpha) \in k(\alpha)$ denote the image of h with respect to the canonical homomorphism $\pi_\alpha : K[V \times \mathbb{A}^1] \to k(\alpha)$. In particular, we set

$$x(\alpha) := (x_1(\alpha), \ldots, x_m(\alpha)) \in \mathbb{A}^m(k(\alpha)).$$

Now let $R \supset K$ be a real closure of K, $z \in V(R)_{\text{reg}} \times \mathbb{A}^1(R)$ and $\varepsilon \in K^*$. Let $\mathfrak{m}_z \subset K[V \times \mathbb{A}^1]$ be the maximal ideal of z. Choose generators $f_1, \ldots, f_k \in K[V \times \mathbb{A}^1]$ of $\mathfrak{m}_z$ and let

$$f_0 := \sum_{i=1}^{k} f_i^2.$$

In $\operatorname{Sper} K[V \times \mathbb{A}^1]$ we consider the set

$$C := \{\alpha \,|\, k(\alpha) \models \exists z' \colon f_0(z') = 0 \wedge ||z' - (x(\alpha), T(\alpha))|| < \varepsilon^2\}.$$

Then, C is constructible, since the theory of real closed fields admits elimination of quantifiers. We claim that C is even open constructible. To this end it is sufficient to show that C is closed under generalizations [6, Corollaire 7.1.21]. So let $\beta \in C$ and let $\alpha \subset \beta$. Then, there are a real closed extension $k(\alpha\beta) \supset k(\beta)$ and a place

$$\lambda \, : \, k(\alpha) \to k(\alpha\beta) \cup \infty$$

such that β is induced by the homomorphism

$$\lambda \circ \pi_\alpha \, : \, K[V \times \mathbb{A}^1] \to K[V \times \mathbb{A}^1]/\operatorname{supp}(\alpha) \to k(\alpha\beta)$$

[6, Proposition 10.2.3]. The valuation ring $B_\lambda \subset k(\alpha)$ of λ is Henselian as $k(\alpha)$ is real closed. Hence, there is a subfield $F \subset k(\alpha)$ such that λ induces an isomorphism between F and $k(\beta)$. Let $a_1, \ldots, a_m, b \in F$ such that

$$(a_1, \ldots, a_m, b) = (\lambda^{-1}(x_1(\beta)), \ldots, \lambda^{-1}(x_m(\beta)), \lambda^{-1}(T(\beta))).$$

Since $\beta \in C$ we find $z' \in \mathbb{A}^{m+1}(F)$ with

$$f_0(z') = 0 \wedge ||z' - (a_1, \ldots, a_m, b)|| < \varepsilon^2.$$

Since $x_i(\alpha) - a_i$ and $T(\alpha) - b$ are infinitesimally small with respect to F we still have

$$||z' - (x(\alpha), T(\alpha)|| < \varepsilon^2.$$

Hence $\alpha \in C$ which shows that C is open. Consequently, $B := \operatorname{Sper} K[V \times \mathbb{A}^1] \setminus C$ is closed constructible and $f_0(\alpha) \neq 0$ for all $\alpha \in B$. By the general "Łojasiewicz inequality" [7, Satz 9, p. 145] there are $n \in \mathbb{N}$ and $h \in K[V \times \mathbb{A}^1]$ such that

$$(*) \quad (1 + \varepsilon^2)^{2n}(\alpha) \leq (1 + h(\alpha)^2) f_0(\alpha)^2$$

for all $\alpha \in B$. Now let $f := (1 + h^2) f_0^2$. Then $f(z) = 0$ where $z \in V(R)_{\mathrm{reg}} \times \mathbb{A}^1(R)$ is the element we started with. By (3) we find $y \in V(K)_{\mathrm{reg}} \times \mathbb{A}^1(K)$ and a total order $Q \subset K$ with

$$f^2(y) <_Q \varepsilon^2.$$

Let $\alpha \in \operatorname{Sper} K[V \times \mathbb{A}^1]$ be the ordering induced by the homomorphism

$$K[V \times \mathbb{A}^1] \to (K, Q) : (x, T) \mapsto y.$$

Then $f(\alpha)^2 < \varepsilon^2$. Now $(*)$ implies $\alpha \in C$. Hence, we find $z' \in \mathbb{A}^{m+1}(k(\alpha))$ such that

$$(**) \quad k(\alpha) \models f_0(z') = 0 \wedge ||z' - (x(\alpha), T(\alpha))|| < \varepsilon.$$

Let $Q' \subset K(z')$ be the total order induced by $k(\alpha)$. It follows from the definition of f_0 that the maximal ideal $\mathfrak{m}_{z'}$ is just $\mathfrak{m}_z$. Therefore, z and z' are conjugated over K. Let $Q'' \subset K(z)$ be the image of Q' under the corresponding conjugation. Then $(**)$ implies

$$||z - y|| <_{Q''} \varepsilon^2$$

which shows (4).

(4) $\Rightarrow$ (1): Let

$$f \in K[V] \setminus \sum K(V)^2.$$

Then there are a real closure R of K and $z \in V(R)_{\mathrm{reg}}$ with $f(z) < 0$. In particular, $L := K(z, \sqrt{-f(z)})$ is a real field. By Corollary 4.4 there is some $\varepsilon \in K^*$ such that for all total orders $Q \subset L$ we have

$$(***) \quad (L, Q) \models \forall\, y = (y_1, \ldots, y_{m+1}) : ||(z, \sqrt{f(z)}) - y|| <_Q \varepsilon^2$$
$$\Rightarrow f(z) \cdot f(y) >_Q 0.$$

By (4) we find $y \in V(K)_{\mathrm{reg}} \times \mathbb{A}^1(K)$ and a total order $Q \subset L$ with

$$||(z, \sqrt{-f(z)}) - y|| <_Q \varepsilon^2.$$

Since $f(z) <_Q 0$, $(***)$ now implies $f(y) <_Q 0$. Therefore, $1 \in T(V|K)$. $\square$

In the next step, we will deduce from the last result a geometric description of the pairs $V|K$ with $1 \in T(V|K)$. Given any variety V over K we let

$$V(K)^{\geq} := \{\alpha \in \operatorname{Sper} K[V] \mid K[V]/\operatorname{supp}(\alpha) = K\}$$

denote the subspace of the “K-rational points” of $\operatorname{Sper} K[V]$ and

$$V(K)^{\geq}_{\mathrm{reg}} := \{\alpha \in V(K)^{\geq} \mid \operatorname{supp}(\alpha) \text{ is regular}\}$$

the subspace of the “K-rational regular points” of $\operatorname{Sper} K[V]$. Finally, let

$$\operatorname{Cent}(V) \subset \operatorname{Sper} K[V]$$

be the closure of the real spectrum $\operatorname{Sper} K(V)$ of the function field of V.

Theorem 4.6. *Let V be a variety over K. Then the following statements are equivalent*:

(1) $T(V|K) \neq \emptyset$.
(2) $1 \in T(V|K)$.
(3) $V(K)^{\geq}_{\mathrm{reg}}$ *is dense in* $\operatorname{Cent}(V)$.

Proof. The equivalence of (1) and (2) was shown in [3, Proposition 1.6]. We first prove (2) $\Rightarrow$ (3). Let

$$K[V] = K[x_1, \ldots, x_m] = K[X_1, \ldots, X_m]/(g_1, \ldots, g_l)$$

and $f_1, \ldots, f_k \in K[V]$ such that

$$D(f_1, \ldots, f_k) := \{\alpha \in \operatorname{Cent} \mid f_1, \ldots, f_k \in \alpha \setminus \operatorname{supp}(\alpha)\} \neq \emptyset.$$

Pick any $\alpha \in D(f_1, \ldots, f_k) \cap \operatorname{Sper} K(V)$. Then $x = (x_1, \ldots, x_m) \in V(k(\alpha))_{\mathrm{reg}}$ and

$$k(\alpha) \models f_1(x) > 0 \wedge \cdots \wedge f_k(x) > 0.$$

Let $R \subset k(\alpha)$ be the relative algebraic closure of K in $k(\alpha)$. By the model completeness of the theory of real closed fields we find $a \in V(R)_{\mathrm{reg}}$ with

$$(*) \quad R \models f_1(a) > 0 \wedge \cdots \wedge f_k(a) > 0.$$

Let

$$L = K(a, \sqrt{f_1(a)}, \ldots, \sqrt{f_k(a)}) \subset R.$$

For each $i \in \{1, \ldots, k\}$, there is by Corollary 4.4 some $\delta_i \in K$ such that for all $\beta \in \operatorname{Sper} L$ we have

$$k(\beta) \models \forall x \in \mathbb{A}^m(k(\beta)) : ||x - a|| < \delta_i^2 \Rightarrow f_i(x) \cdot f_i(a) > 0.$$

Let

$$\delta := \prod_{1=1}^{k} \frac{\delta_i}{1 + \delta_i^2}.$$

Since $f_i(a) \in L^2$ for all $1 = 1, \ldots, n$, then every $\beta \in \operatorname{Sper} L$ even satisfies

$$(**) \quad k(\beta) \models \forall\, x \in \mathbb{A}^m(k(\beta)) : ||x - a|| < \delta^2 \Rightarrow \bigwedge_{i=1}^{k} f_i(x) > 0.$$

By Proposition 4.5 (4) we find $y \in V(K)_{\mathrm{reg}}$ and $\beta \in \operatorname{Sper} L$ with $k(\beta) \models ||y-a|| < \delta^2$. Now $(**)$ implies

$$k(\beta) \models f_1(y) > 0 \wedge \cdots \wedge f_k(y) > 0.$$

Let $\gamma \in V(K)^{\geq}_{\mathrm{reg}}$ be the ordering which corresponds to the homomorphism

$$\varphi : K[V] = K[x_1,\ldots,x_m] \to R : x_i \mapsto y_i.$$

Then $\gamma \in D(f_1,\ldots,f_k)$. It remains to prove (3) $\Rightarrow$ (2). Assume that $f \in K[V]$ is not a sum of squares in the function field $K(V)$. Then $D(-f) \subset \operatorname{Cent}(V)$ is not empty. Since $V(K)^{\geq}_{\mathrm{reg}}$ is dense in $\operatorname{Cent}(V)$ there are some $z \in V(K)_{\mathrm{reg}}$ and a total order $Q \subset K$ with $f(z) <_Q 0$. In other words, $f(z) \notin \sum K^2$. Hence, $1 \in T(V|K)$. $\square$

If V is a smooth variety, then $V(K)^{\geq}_{\mathrm{reg}} = V(K)^{\geq}$ and $\operatorname{Cent}(V) = \operatorname{Sper} K[V]$. Hence we get

Corollary 4.7. *Let V be an irreducible smooth variety over K. Then the following statements are equivalent*:

(1) $T(V|K) \neq \emptyset$.
(2) $1 \in T(V|K)$.
(3) $V(K)^{\geq}$ *is dense in* $\operatorname{Sper} K[V]$.

Finally, let us have a look at locally dense fields. By [11, Theorem 3] a field K is locally dense if and only if $1 \in T(\mathbb{A}^d|K)$ for some $d \in \mathbb{N}$ if and only if $1 \in T(\mathbb{A}^d|K)$ for all $d \in \mathbb{N}$. Therefore, the last result implies

Corollary 4.8. *For a field K the following statements are equivalent*:

(1) *K is locally dense*
(2) $\mathbb{A}^d(K)^{\geq}$ *is dense in* $\operatorname{Sper} K[X_1,\ldots,X_d]$ *for some* $d \in \mathbb{N}$.
(3) $\mathbb{A}^d(K)^{\geq}$ *is dense in* $\operatorname{Sper} K[X_1,\ldots,X_d]$ *for all* $d \in \mathbb{N}$.

Let V be a variety over K and assume $1 \in T(V|K)$. If $\dim V \geq 2$, then Corollaries 3.4 imply $1 \in T(\mathbb{A}^d|K)$ for all $d \in \mathbb{N}$. In the next step we will show that the same holds for arbitrary varieties.

Corollary 4.9. *Let V be a variety over K with $1 \in T(V|K)$. Then $1 \in T(\mathbb{A}^d|K)$ for all $d \in \mathbb{N}$.*

Proof. Let $d \in \mathbb{N}$. Then an iterated application of Proposition 4.5 shows $1 \in T(V \times \mathbb{A}^d|K)$. Now let

$$f \in K[T_1,\ldots,T_d] \setminus \sum K(T_1,\ldots,T_d)^2.$$

As in the proof of Corollary 3.4 we see

$$f \notin \sum K(V \times \mathbb{A}^d)^2 = \sum K(V)(T_1,\ldots,T_d)^2.$$

Hence, there are $z_1,\ldots,z_d \in K$ with $f(z_1,\ldots,z_d) \notin \sum K^2$. $\square$

Let (K,v) be a real Henselian-valued field and assume that there is a variety V over K with $1 \in T(V|K)$. Then, the value group of v is divisible as has been shown in [3, Theorem 2.6]. In view of Proposition 4.5 we can now sharpen this result as follows.

Proposition 4.10. *Let K be a real field which admits a nontrivial Henselian valuation v. Then the following statements are equivalent*:

(1) *K is real closed.*
(2) *$1 \in T(V|K)$ for all varieties V over K.*
(3) *$1 \in T(V|K)$ for some variety V over K.*
(4) *$1 \in T(\mathbb{A}^1|K)$.*

Proof. The implication (1) $\Rightarrow$ (2) is well known. Clearly (2) $\Rightarrow$ (3) and by Corollary 4.9 we know (3) $\Rightarrow$ (4). So assume $1 \in T(\mathbb{A}^1|K)$. Then the value group of v is divisible as has been shown in [3, Theorem 2.6]. Therefore, it remains to prove that the residue field K_v of v is real closed. Since v is Henselian there is a subfield $L \subset K$ which isomorphic to K_v under the canonical place

$$\lambda_v : K \to K_v \cup \infty$$

associated with v. Assume by way of contradiction that L is not real closed. Then there exists a proper real extension $K(z) \supsetneq K$ with z algebraic over L. Let w be the unique extension of v to $K(z)$. Then w is Henselian as well. Let $y \in K$. Then, $w(z-y) \leq 0$ as $w(z)=0$ and $z \notin L \cong K_v$. Now pick $\varepsilon \in K^*$ with $v(\varepsilon) > 0$. Then

$$(*) \quad w(z-y) < w(\varepsilon^2) \quad \text{for all } y \in K.$$

Let $Q \subset K(z)$ be any total order. Since w is Henselian, w is compatible with Q. Hence, $(*)$ implies $||z-y|| >_Q \varepsilon^2$. Now, Proposition 4.5(4) gives the contradiction $1 \notin T(\mathbb{A}^1|K)$. $\square$

We conclude this section with another useful consequence of Proposition 4.5.

Proposition 4.11. *Let $V|K$ be a variety with $1 \in T(V|K)$ and let $L \supset K$ be a finite algebraic extension. Then $1 \in T(V \otimes_K L|L)$.*

Proof. Let $W := V \otimes_K L$. By Lemma 3.3, $V(K)_{\text{reg}}$ is not empty which implies that V is absolutely irreducible. Hence, W is an irreducible variety over L. Pick $\theta \in L$ with $L = K[\theta]$. We show that $W|L$ satisfies the condition (Proposition 4.5(4)). So let R be a real closure of L, $z \in W(R)_{\text{reg}} \times \mathbb{A}^1(R)$ and $\varepsilon \in L^*$. Given any total order $Q \subset L$ there is some $\varepsilon_Q \in K^*$ such that

$$0 <_Q \varepsilon_Q^2 <_Q \varepsilon^2.$$

By the compactness of $\operatorname{Sper} L$ there are finitely many $\varepsilon_1, \dots, \varepsilon_k \in K^*$ such that

$$\operatorname{Sper} L = \bigcup_{i=1}^{k} \{Q \in \operatorname{Sper} L \mid 0 <_Q \varepsilon_i^2 <_Q \varepsilon^2\}.$$

Let

$$\varepsilon_0 := \prod_{i=1}^{k} \frac{\varepsilon_i}{1+\varepsilon_i^2}.$$

Then $0 <_Q \varepsilon_0^2 <_Q \varepsilon^2$ for all $Q \in \operatorname{Sper} L$. Therefore, it is sufficient to show that there are $y \in W(L)_{\mathrm{reg}} \times \mathbb{A}^1(L)$ and a total order $Q \subset L(z)$ with

$$||z - y|| <_Q \varepsilon_0^2.$$

By assumption $1 \in T(V|K)$. Applying twice Proposition 4.5 we get $1 \in T(V \times \mathbb{A}^2|K)$. By Proposition 4.5(4) we then find $(y_V, y_1, y_2) \in V(K)_{\mathrm{reg}} \times \mathbb{A}^2(K)$ and a total order $Q \subset K(z, \theta) = L(z)$ such that

$$||(z,\theta) - (y_V, y_1, y_2)|| <_Q \varepsilon_0^2.$$

Since $W(L)_{\mathrm{reg}} = V(L)_{\mathrm{reg}}$ (as subsets of some affine space over L), we have $(y_V, y_1) \in W(L)_{\mathrm{reg}} \times \mathbb{A}^1(L)$ and obviously

$$||z - (y_V, y_1)|| <_Q \varepsilon_0^2.$$

Now apply Proposition 4.5(4). □

5. Semigroups realized by $T(V|K)$

Let $\mathbb{P} \subset \mathbb{N}$ be the set of prime numbers. Given $\mathbb{P}_0 \subset \mathbb{P}$ we let $\langle \mathbb{P}_0 \rangle \subset \mathbb{N}$ denote the multiplicative semigroup with 1, generated by $\mathbb{P}_0$. By [3, Corollary 2.4] there exists for every $\mathbb{P}_0 \subset \mathbb{P}$ a field K such that

$$S(V|K) = \langle \mathbb{P}_0 \rangle$$

for all varieties over K. But in the case of $T(V|K)$ we have so far only examples for

$$T(V|K) = \emptyset, \quad T(V|K) = \{1\} \quad \text{and} \quad T(V|K) = \mathbb{N}.$$

It is the goal of this section to show that for every $\mathbb{P}_0 \subset \mathbb{P}$ with $2 \in \mathbb{P}_0$ there exists a field K such that

$$T(\mathbb{A}^d|K) = \langle \mathbb{P}_0 \rangle$$

for all $d \in \mathbb{N}$.

Lemma 5.1. *Let $q \in \mathbb{N}$ be an odd prime number and $K[a] \supset K$ an algebraic extension with $a^q \in K \setminus K^q$. Then*

$$\sum K[a]^{2p} \cap K = \sum K^{2p}$$

for all primes $p \neq q$.

Proof. Let $L{:=}K[a]$. Since q is odd every total order of K uniquely extends to a total order of L. Hence, every real valuation of K uniquely extends to a real valuation of L. Therefore, it is sufficient to show

$$pv(L) \cap v(K) = pv(K)$$

for all real valuations v of L. Thereby, we have to distinguish several cases. First, assume $v(a^q) \notin qv(K)$. Since q is prime this implies $[v(L) : v(K)] = q$. But then

$$pv(L) \cap v(K) = pv(K)$$

for all primes $p \neq q$. Next, assume $v(a^q) = qv(b)$ for some $b \in K$. Let $c{:=}ab^{-1}$. Then, $v(c) = 0$ and $L = K[c]$. We let L_v and K_v denote the residue fields of v and $v|K$. Let,

$$\lambda : L \to L_v \cup \infty$$

be the canonical place associated with v. If $\lambda(c^q) \notin K_v^q$, then $[L_v : K_v] = q = [L : K]$ which implies $v(L) = v(K)$. So it remains to consider the case $\lambda(c^q) \in K_v^q$. But then again $v(L) = v(K)$ as (L, v) now embeds into the henselization of K with respect to $v|K$. $\square$

Let $\mathscr{L} \subset \mathbb{P}$ be nonempty. For F any field, q a prime number and $A_q \subset F$ a system of representants of $F^*/F^{*q} \setminus \{1\}$, we set

$$F^{\mathscr{L}}{:=}F(\sqrt[q]{a} \mid a \in A_q, q \in \mathscr{L}).$$

Given a field K we define its $\mathscr{L}$-radical closure as follows. First set

$$K_1{:=}K^{\mathscr{L}}.$$

If K_n has already been defined for some $n \in \mathbb{N}$ let

$$K_{n+1}{:=}K_n^{\mathscr{L}}.$$

Then,

$$K(\mathscr{L}){:=}\bigcup_{n \in \mathbb{N}} K_n$$

is called the $\mathscr{L}$-radical closure of K. We will need the following property of $\mathscr{L}$-radical closures.

Lemma 5.2. *Let $\mathscr{L} \subset \mathbb{P}$ be given with $2 \notin \mathscr{L}$ and let $K \subset L \subset K(\mathscr{L})$ be any intermediate field. Then*

$$\sum K(\mathscr{L})^{2p} \cap L = \sum L^{2p}$$

for every $p \in \mathbb{P} \setminus \mathscr{L}$.

Proof. Assume by way of contradiction that there is some $a \in \sum K(\mathscr{L})^{2p} \cap L$ with $a \notin \sum L^{2p}$. Then there are a finite sequence $L = L_0 \subset L_1 \subset \cdots \subset L_n \subset K(\mathscr{L})]$ of field extensions, $q_0, \ldots, q_{n_1} \in \mathscr{L}$ and $a_i \in L_i \setminus L_i^{q_i}$ such that

$$L_{i+1} = L_i(\sqrt[q_i]{a_i}) \quad \text{and} \quad a \in \sum L_n^{2p}.$$

But then an iterated application of Lemma 5.1 implies the contradiction $a \in \sum L^{2p}$. $\square$

We can now prove the main result of this section.

Theorem 5.3. *Let $\mathbb{P}_0 \subset \mathbb{P}$ be a set of prime numbers with $2 \in \mathbb{P}_0$. Then, there is a field K such that for all $d \in \mathbb{N}$*

$$T(\mathbb{A}^d|K) = \langle \mathbb{P}_0 \rangle.$$

Proof. Let $F := \mathbb{Q}(X, Y)$ be a purely transcendental extension, $\mathscr{L} := \mathbb{P} \setminus \mathbb{P}_0$ and let $K := F(\mathscr{L})$. We claim

$$T(\mathbb{A}^d|K) = \langle \mathbb{P}_0 \rangle.$$

The construction of K implies $K^2 = K^{2q}$ for all $q \in \mathscr{L}$. Hence, $T(\mathbb{A}^d|K) \subset \langle \mathbb{P}_0 \rangle$. For the converse direction let $p \in \mathbb{P}_0$ and

$$f \in K(T_1, \ldots, T_d) \Big\backslash \sum K(T_1, \ldots, T_d)^{2p}.$$

Then, $f \in L(T_1, \ldots, T_d)$ for some finite extension L of $\mathbb{Q}(X, Y)$. Since $\mathrm{tr.d.}(L|\mathbb{Q}) = 2$ we get $p \in T(\mathbb{A}^d|L)$ by Corollary 2.7. Hence there is some $x \in \mathbb{A}^d(L)$ with $f(x) \notin \sum L^{2p}$. From Lemma 5.2 we infer $f(x) \notin \sum K^{2p}$ and therefore $p \in T(\mathbb{A}^d|K)$. Now [3, Theorem 1.8] implies $\langle \mathbb{P}_0 \rangle \subset T(\mathbb{A}^d|K)$. $\square$

By obvious reasons it is essentially harder to investigate the semigroup $T(V|K)$ rather than $S(V|K)$. But nevertheless, one might wonder whether there is a certain relationship between the semigroups $T(V|K)$ and $S(V|K)$. For example, in view of the results of Section 4 one might ask whether $1 \in T(V|K)$ and $p \in S(V|K)$ imply $p \in T(V|K)$. Our final result shows that this fails.

Proposition 5.4. *Let $\mathbb{Q}(T)$ be a simple transcendental extension of $\mathbb{Q}$ and let $d \in \mathbb{N}$. Then*

$$S(\mathbb{A}^d|\mathbb{Q}(T)) = \mathbb{N} \quad \textit{but } T(\mathbb{A}^d|\mathbb{Q}(T)) = \{1\}.$$

Proof. Let $K = \mathbb{Q}(T)$. By Theorem 2.1 and [3, Theorem 1.11] we know $S(\mathbb{A}^d|K) = \mathbb{N}$ and $1 \in T(\mathbb{A}^d|K)$. Let $p \in \mathbb{N}$ be any prime number. Note that $T(\mathbb{A}^d|K) \subset T(\mathbb{A}^1|K)$. Hence, it is sufficient to show $p \notin T(\mathbb{A}^1|K)$. To this end pick a prime number $q \in \mathbb{N}$ with $q \neq 2, p$ and consider the polynomial

$$f(X) = (X^{pq} - 2)^2 + (T - 2T^p)^{2q} \in K[X].$$

We claim $f(X) \notin \sum K(X)^{2p}$ but $f(y) \in \sum K^{2p}$ for all $y \in K$. In order to show the first claim choose $\alpha \in \mathbb{R}$ with $\alpha^{pq} = 2$. Then,

$$f(\alpha) = (T - 2T^p)^{2q}.$$

Let v be the unique real valuation of $K(\alpha)$ with $v(T) > 0$. Then,

$$v(f(\alpha)) = 2qv(T) \notin 2pv(K(\alpha)).$$

Hence, $f(\alpha) \notin \sum K(\alpha)^{2p}$ which implies $f(X) \notin \sum K(X)^{2p}$. Next let $y \in K$. It remains to prove $f(y) \in \sum K^{2p}$. Thus, given any nontrivial valuation v of K we have to show $v(f(y)) \in 2pv(K)$. Let $\mathbb{R}_0$ be the real closure of $\mathbb{Q}$. Since v is nontrivial, either v is the degree valuation or there exists $\alpha \in \mathbb{R}_0$ such that v extends to a valuation w of $\mathbb{R}_0(T)$ with $w(T-\alpha) > 0$. We treat these cases separately. First assume that v is the degree valuation. If $v(y) \geq 0$, then

$$v(f(y)) = v((T - 2T^p)^{2q}) = 2pqv(T).$$

If $v(y) < -1$, then obviously $v(f(y)) = 2pqv(y)$. So assume $v(y) = -1$. Then there are $0 \neq a \in \mathbb{Q}$ and $z \in K$ with $v(z) \geq 0$ and $y = aT + z$. Hence,

$$v(f(y)) = v((aT)^{2pq} + 2^{2q}T^{2pq}) = 2pqv(T).$$

Next, we treat the case that there is some $\alpha \in \mathbb{R}_0$ such that v extends to the unique valuation w of $K(\alpha)$ with $w(T-\alpha) > 0$. If $v(y) < 0$, then $v(f(y)) = 2pqv(y)$. So assume $v(y) \geq 0$. We claim that this implies $w(f(y)) = 0$. Let

$$\lambda : K(\alpha) \to \mathbb{Q}(\alpha) \cup \infty$$

be the canonical place associated with w. Since $v(y) \geq 0$ there are $b \in \mathbb{Q}(\alpha)$ and $z \in K(\alpha)$ with $w(z) \geq 0$ and $y = b + (T-\alpha)z$. Then,

$$\lambda(f(y)) = (b^{pq} - 2)^2 + (\alpha - 2\alpha^p)^{2q}.$$

Assume by way of contradiction that $\lambda(f(y)) = 0$. Then,

$$b^{pq} - 2 = 0 \quad \text{and} \quad \alpha - 2\alpha^p = 0$$

which implies $\alpha=0$ or $\alpha^{1-p}=2$. If $\alpha=0$, then $\mathbb{Q}(\alpha)=\mathbb{Q}$ and, consequently, $b^{2pq}-2 \neq 0$. On the other hand, if $\alpha^{1-p}=2$, then 2 is not a pth power in $\mathbb{Q}(\alpha)$ which implies again the contradiction $b^{pq} - 2 \neq 0$. Hence $\lambda(f(y)) \neq 0$ which shows $v(f(y)) \in 2pv(K)$. Therefore, $p \notin T(\mathbb{A}^1|K)$. $\square$

References

[1] E. Becker, Summen n-ter Potenzen in Körpern, J. Reine Angew. Math. 307/308 (1979) 8–30.

[2] E. Becker, The real holomorphy ring and sums of 2nth powers, in: J.-L. Colliot-Thélène, M. Coste, L. Mahé, M.-F. Roy (Eds.), Géométrie Algébrique Réelle et Formes Quadratiques, Lecture Notes in Mathematics, vol. 959, Springer, Berlin, 1982, pp. 139–181.

[3] E. Becker, R. Berr, F. Delon, D. Gondard, Hilbert's 17th problem for sums of 2nth powers, J. Reine Angew. Math. 450 (1994) 139–157.

[4] R. Berr, The p-real spectrum of a commutative ring, Comm. Algebra 20 (1992) 3055–3103.

[5] R. Berr, Null- and Positivstellensätze for generalized real closed fields, J. Pure Appl. Algebra 125 (1998) 19–53.

[6] J. Bochnak, M. Coste, M.-F. Roy, Géométrie Algébrique Réelle, Springer, Berlin, 1987.

[7] M. Knebusch, C. Scheiderer, Einführung in die reelle Algebra, Vieweg, Braunschweig, 1989.

[8] K. McKenna, New facts about Hilbert's seventeenth problem, Lecture Notes in Mathematics, vol. 498, Springer, Berlin, 1975, pp. 220–230.

[9] A. Prestel, Sums of squares over fields, Proceedings of the 5th School of algebra, Rio de Janeiro, 1978, Soc. Brasil. Mat., Rio de Janeiro, 1978, pp. 33–44.
[10] N. Schwartz, Chain signatures and real closures, J. Reine Angew. Math. 347 (1984) 1–20.
[11] Zeng Guangxin, A characterization of preordered fields with the weak Hilbert property, Proc. AMS 104 (1988) 335–342.